Metric and Topdogical Spaces

1 Decide if the function $d(x,y) = \{0 \text{ if } x=y,\ x+y \neq y\}$ is or is not a metric on the set $\mathbb{N}_{\geq 1}$, on $\mathbb{N}_{\geq 0}$, on $\mathbb{N}_{\geq 0} \cup \{-1\}$, on $\mathbb{R}_{\geq 0}$. If "yes" at least once, then draw the respective open disks $B_{r=10}(x_0=2)$ and $B_{r=2}(x_0=10)$

For d to be a metric, it needs to satisfy the 3 axioms. On the set $\mathbb{N}_{\geq 1}$, we see that

M_1) holds, as $d(x,y) \geq 0$ for all $x,y \in \mathbb{N}_{\geq 1}$, as $x \neq y$ $x,y \geq 0$, so $x+y>0$, and when $x=y$, by definition of d, $d(x,y)=0$
$\Rightarrow d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$.

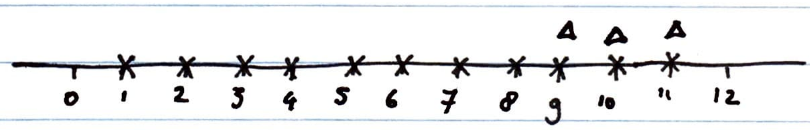
M_2) also holds, as $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ x+y & \text{if } x \neq y. \end{cases}$

~~On the set $\mathbb{N}_{\geq 0}$,~~

Then, we see that if $y=x$, $d(y,x)=0$, and $x+y=y+x$ so, $d(y,x)=y+x=x+y$ if $y \neq x$
$\Rightarrow d(y,x) = d(x,y)$ for all $x,y \in \mathbb{N}_{\geq 1}$

M_3) Let us take any $x,y,z \in \mathbb{N}_{\geq 1}$. Then, consider 2 cases. case 1) $x=z$. Then $d(x,z)=0 \leq d(x,y)+d(y,z)$ regardless of $y=x=z$, or $y \neq x=z$.

case 2) $x \neq z$. Then $d(x,z) = x+z < x+y+y+z$ $= d(x,y)+d(y,z)$ for all $y \in \mathbb{N}_{\geq 1}$ as $y \neq 0$.



$\times = B_{r=10}(x_0=2)$
$\Delta = B_{r=2}(x_0=10)$

Next, we consider $\mathbb{N}_{\geq 0}$. Note that for M_1, nothing changes, as $x \neq y$, but 1 of them being 0, means $d(x,y) = x+y = x+0 = x > 0$
for example $y=0$
and for $x=y=0$ $d(x,y)=0$ by definition.

For M_2), the argument holds as it is.

For M_3), If case 1), we see that the argument as it is holds for case 2), note that $d(x,z) = x+z \leq x+y+y+z = d(x,y) + d(y,z)$ as y can equal 0 now.

Continuation of question 1:

But, we still have $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathbb{N}_{\geq 0}$.
For $B_{r=10}(x_0=2)$, we add an x to the point $\{0\}$.
For $B_{r=2}(x_0=10)$, we don't add anything

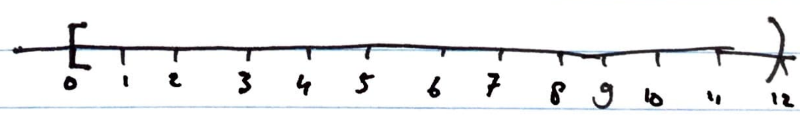
Now, we consider $\mathbb{N}_{\geq 0} \cup \{-1\}$.
Note that for M_1), we get that if $x=1$, $y=-1$,
$d(x,y) = x+y = 1-1 = 0$ even though $x \neq y$. As this axiom doesn't hold, d is not a metric on $\mathbb{N}_{\geq 0}\{-1\}$.

Lastly, we consider $\mathbb{R}_{\geq 0}$.
For M_1), consider
$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ x+y & \text{if } x \neq y \end{cases}$$
From this we see, as x,y nonnegative, that $d(x,y) \geq 0$, and only 0 if $x=y$ ($x+y$ is zero iff $y=-x$, but we don't have negative numbers). Trivially, if $x=y \Rightarrow d(x,y)=0$ by definition.

For M_2)
Consider any $x,y \in \mathbb{R}_{\geq 0}$
$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ x+y & \text{if } x \neq y \end{cases}$$
$\Rightarrow 0$ if $y=x$
$x+y = y+x$
$$\Rightarrow d(y,x) = \begin{cases} 0 & \text{if } y=x \ (x=y) \\ y+x = x+y & \text{if } y \neq x \end{cases}$$
so $d(y,x) = d(x,y)$

For M_3), the same argument as for $\mathbb{N}_{\geq 0}$ holds

$B_{r=10}(x_0=2)$



$B_{r=2}(x_0=10)$

2. Let (X, d_X), (Y, d_Y) be metric spaces. Prove that a map $f: X \to Y$ is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all subsets $A \subseteq X$

We solved this exercise in homework 3.
$\Rightarrow$ Let us assume f is continuous. Let A be any subset of X, and let $y \in f(\bar{A})$. We know that there must exist an $x \in \bar{A}$ s.t. $f(x) = y \in f(\bar{A})$. As f is continuous, we know that for $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(y)$ by proposition 5.30.
By definition of the closure, $\exists a \in A$ s.t. $a \in A \cap B_\delta(x)$
This gives us: $f(a) \in f(B_\delta(x)) \subseteq B_\varepsilon(y)$. This means that $f(a) \in B_\varepsilon(y) \cap f(A)$. This gives that $f(a) \in \overline{f(A)}$

$$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$$

$\Leftarrow$ Let us assume that $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. From proposition 6.6, we know that f is continuous if $f^{-1}(V)$ is closed in X whenever V is closed in Y. Suppose we have any $V \subseteq Y$ closed in Y. Then, we know by proposition 6.11c that $V = \bar{V}$. Note that $f f^{-1}(V) \subseteq V$.
By proposition 6.11 b), we now see that

$$\overline{f(f^{-1}(V))} \subseteq \bar{V} = V. \qquad *$$

By our assumption, we also have

$$f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \qquad \Delta$$

Combining $*$ and Δ gives us

$$f(\overline{f^{-1}(V)}) \subseteq \bar{V} = V \Rightarrow \overline{f^{-1}(V)} \subseteq f^{-1}(V).$$

Now, by proposition 6.11 a), we have that

$f^{-1}(V) \subseteq \overline{f^{-1}(V)}$, which means that $\overline{f^{-1}(V)} = f^{-1}(V)$
We know that $\overline{f^{-1}(V)}$ is closed in X, so $f^{-1}(V)$ must be closed in X.
So, now by proposition 6.6, as for all V closed in Y, $f^{-1}(V)$ is closed in X, f is continuous.

3 In a space X, its subset V is closed iff it contains its own boundary : $\partial V \subseteq V$

$\Rightarrow$ Assume V is closed. By proposition g.10 C, $V = \overline{V}$.
consider any $x \in \partial V$
By proposition g.20, we know that $\partial V = \overline{V} \cap \overline{X \setminus V}$
$\Rightarrow x \in \partial V = \overline{V} \cap \overline{X \setminus V}$
This means that $x \in \overline{V}$, but as $\overline{V} = V$, $x \in V$.
So $\partial V \subseteq V$.

$\Leftarrow$ Assume that $\partial V \subseteq V$.
We know by proposition g.10 c that V is closed in X iff $V = \overline{V}$
B proposition g.10 a), we know that $V \subseteq \overline{V}$
What remains to be shown is $\overline{V} \subseteq V$.
To this end, consider $v \in \overline{V}$
Then, we can distinguish 2 cases:
case 1) $v \notin \partial V$
case 2) $v \in \partial V$
Let us consider case 1). By definition $\partial V = \overline{V} \setminus \mathring{V}$
$v \notin \partial V = \overline{V} \setminus \mathring{V}$, but $v \in \overline{V} \Rightarrow v \in \mathring{V}$
By proposition g.17 a), $\mathring{V} \subseteq V \Rightarrow v \in V$.

Next, case 2). $v \in \partial V$. We know from proposition g.20 that $\partial V = \overline{V} \cap \overline{X \setminus V}$.
$v \in \partial V = \overline{V} \cap \overline{X \setminus V} \Rightarrow v \in \overline{V}$, $v \in \overline{X \setminus V}$
$v \in \overline{V} \Rightarrow \exists U_1 \subseteq X$ open s.t. $v \in U_1$ $V \cap U_1 \neq \emptyset$
$v \in \overline{X \setminus V} \Rightarrow \exists U_2 \subseteq X$ open s.t. $v \in U_2$, $(X \setminus V) \cap U_2 \neq \emptyset$

4. Give an example of a sequence of open connected subsets $U_n \subseteq \mathbb{E}^2$ of the plane such that $U_n \supseteq U_{n+1}$ for each $n \in \mathbb{N}$ but the intersection $\bigcap_{n=1}^{\infty} U_n$ is not connected.

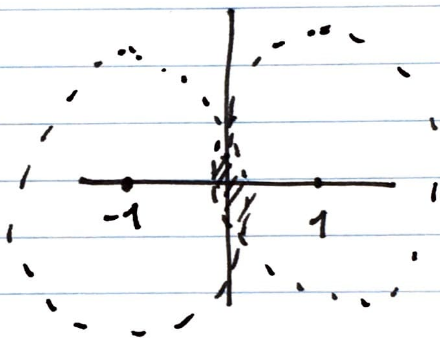
Consider in $\mathbb{E}^2$ the following intersection:

$$B_{1+\frac{1}{n}}(1) \cap B_{1+\frac{1}{n}}(-1) = U_n$$

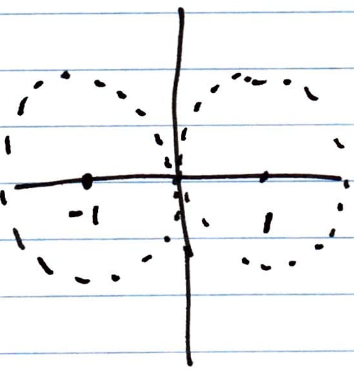
Then $U_n \supseteq U_{n+1}$ U_n are open connected subsets, but note that $\bigcap_{n=1}^{\infty} U_n$ is not connected

$n = 1$

$n = 10$



$n \to \infty$



where we can make the partition $B_1(1) \sqcup B_1(-1)$

($B_1(1) \cup B_1(-1) = U_n$ as $n \to \infty$
$B_1(1) \cap B_1(-1) = \emptyset$
$B_1(1)$, $B_1(-1)$ both open
$B_1(1)$, $B_1(-1)$ both nonempty)

5 Let X be a Hausdorff space
a) In X, every compact is closed.
b) For any point $x \in X$, the intersection of all open subsets of X containing x is the singleton set $\{x\}$.
c) Given an example of topological space X which is not Hausdorff, its topology is not ~~fin~~ cofinite, still b) holds.

a) Let $A \subseteq X$ be compact. (X, T_X)
We know from proposition 11.7 a) that ~~...~~ A is also Hausdorff.
We want to show that ~~for~~ $X \setminus A$ is open in X, i.e,
$\forall x \in X \setminus A,\ \exists U_x \in T_X$ s.t. $x \in U_x \subseteq X \setminus A$

Let us fix any $x \in X \setminus A$, and let ~~...~~ $a_i \in A$. $i \in I$
Then, as X is Hausdorff, we know that
$\exists U_{a_i}(x) \in T_X \mid x \in U_{a_i}(x)$, $\exists U_x(a_i) \in T_X \mid a_i \in U_x(a)$ and
$U_{a_i}(x) \cap U_x(a_i) = \phi$.

Note that $\bigcup_{i \in I} U_x(a_i)$ is an open cover for A. Now, as A is compact, we can find a finite subcover of A, i.e.,
$\exists J$ indexing set $|J| < \infty$ s.t. $\bigcup_{j \in J} U_x(a_j) ~~a~~ \supseteq C$.

Now, let us consider $\bigcap_{j \in J} U_{a_j}(x)$. We see that $x \in \bigcap_{j \in J} U_{a_j}(x)$, and as it's a finite intersection of open sets,
$\bigcap_{j \in J} U_{a_j}(x)$ is open in X.

By construction, $A ~~\in~~ \cap \left(\bigcap_{j \in J} U_{a_j}(x)\right) \subseteq \left(\bigcup_{j \in J} U_x(a_j)\right) \cap \left(\bigcup_{j \in J} U_{a_j}(x)\right) = \phi$

Then, we see that $\bigcap_{j \in J} U_{a_j}(x) \in X \setminus A$.

Now, set $U_x = \bigcap_{j \in J} U_{a_j}(x)$. Then, we see that
$\forall x \in X \setminus A$, $\exists U_x$ s.t. $x \in U_x \subseteq X \setminus A$.
Now, by ~~definition~~, proposition 7.2, $X \setminus A$ is open in X
$\Rightarrow A$ is closed in X.

Continuation of question ~~#~~ 5

b) Let us fix $x^* \in X$ for an arbitrary x^*.
Let us consider 2 cases:
Case 1) $\exists$ a set ~~[illegible]~~ $A \subseteq X$ such that it only contains $\{x^*\}$. Then, regardless of with what subsets containing x^* we intersect, ~~[illegible]~~ this intersection will only contain $\{x^*\}$
Case 2) $\nexists$ such a set A as described above (or we cannot find it). Then, we know that X is Hausdorff.
Let us consider any set $U \subseteq X$ containing x^* and assume $\exists y \in X$ s.t. $y \in U$.
at least 1

Let $y_i \in U$ for $i \in \mathcal{I}$ some indexing set ~~and~~ be all elements of U not equal to x^*.

As X is Hausdorff, $\exists$ open sets $U_{x^*}(y_i) \ni y_i$ $U_{y_i}(x^*) \ni x^*$ such that $U_{x^*}(y_i) \cap U_{y_i}(x^*) = \emptyset$.
~~Let us set this intersection as A_i~~
Note that for $\forall z \in U_{x^*}(y_i)$, $z \notin U_{y_i}(x^*) \cap U$, and define this intersection as A_i.
Now consider $\bigcap_{i \in \mathcal{I}} A_i$. This intersection contains x^*, but x^* only by construction.
$\Rightarrow$ the intersection of all open subsets containing x for any point $x \in X$ is the singleton set $\{x\}$.

c) ~~Consider the topology # Euclidean topology on $\mathbb{R}$ obtained from all open sets satisfying $B_\varepsilon(x)$ $x \in \mathbb{R}$, and let us set $\varepsilon = \frac{1}{2}$.~~
Let us consider the Thick Line as defined in lecture 8.
Then, the Thick line is not Hausdorff, it's topology is not finite, and b) holds.

6 A metric space X is complete. $\Leftrightarrow$ any nested sequence of closed disks $V_n := \overline{B_{r_n}}(x_n) = \{y \in X \mid d(x_n, y) \leq r_n\} \supseteq V_{n+1}$ such that their radii $r_n \to 0$ as $n \to \infty$ has a nonempty intersection. Prove $\Leftarrow$

Let us assume that we have any nested sequence of closed disks $V_n := \overline{B_{r_n}}(x_n) = \{y \in X \mid d(x_n, y) \leq r_n\} \supseteq V_{n+1}$ s.t. $r_n \to 0$ as $n \to \infty$

Consider any Cauchy sequence (x_n) in X, and note that by definition of a Cauchy sequence, for $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. whenever $m, n \geq N$, $d(x_n, x_m) < \varepsilon$.

~~{Now let us choose $\varepsilon > 0$. Then, as $r_n \to 0$ as $n \to \infty$, we can choose an r_N s.t. $r_N < \varepsilon$.}~~

From the sequence (x_n), we will construct a nested sequence of closed disks in the following way:
Let us set $r_n = d(x_n, x_{n-1})$ when $n \geq 2$, and let $r_1 = 2r_2$

Note that like this, we will obtain a nested sequence of closed disks $V_n = \overline{B_{r_n}}(x_n)$
As (x_n) is Cauchy, for any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ s.t. whenever $n, m \geq N$, $d(x_n, x_m) < \varepsilon$. Now, let us choose $r_N < \varepsilon$. Then, by construction of V_n, $d(x_n, x_m) \leq r_N < \varepsilon \Rightarrow d(x_n, x_m) < \varepsilon$.
As we can make ε arbitrarily small, we can find $N \in \mathbb{N}$ s.t. $\varepsilon \to 0$ as $N \to \infty$. In other words, we can make $r_N \to$ as $N \to \infty$. Now by assumption $\bigcap_{n=1}^{\infty} V_n$ is not empty. $\Rightarrow \exists\, x \in \bigcap_{n=1}^{\infty} V_n$
This means that, as $(x_n) \subseteq V_1$, (x_n) converges to $x \in \bigcap_{n=1}^{\infty} V_n$.

$\Rightarrow (x_n)$ converges to a point in X

Continuation question 6:
As we choose any Cauchy sequence in X, all Cauchy sequences in X will converge to a point in X, and thus, by definition, X is complete.